1. A RECURRENCE RELATION FOR A FUNCTION OR SEQUENCE IS AN EQUATION EXPRESSING LATER VALUES OF THE FUNCTION OR SEQUENCE IN TERMS OF EARLIER VALUES.

FOR EXAMPLE:
- $f(n+1) = f(n) + 3$
- $B_{n+1} = 2 \cdot B_n$
- $F_{n+2} = F_{n+1} + F_n$.

2. (a) A RECURRENCE RELATION IS LINEAR WHEN IT USES ONLY ADDITION & CONSTANT MULTIPLES IN THE EQUATION.

E.G.:
- $A_{n+2} = 3A_{n+1} - 2A_n$ IS LINEAR
- $B_{n+1} = (n+1)B_n$ IS NOT LINEAR (NON-CONSTANT MULTIPLE!)
- $C_{n+2} = C_{n+1} \cdot C_n$ IS NOT LINEAR (MULTIPLICATION OF TERMS)
- $f(n+1) = f(n) + 3$ IS NOT LINEAR (ADDING A CONSTANT)

(b) THE CHARACTERISTIC POLYNOMIAL OF A LINEAR RECURRENCE RELATION IS WHAT WE GET WHEN WE:

① REPLACE THE n^{TH} TERM (E.G., A_n OR $f(n)$) BY r^n

② COLLECT ALL VARIABLES TO ONE SIDE, AND

③ DIVIDE BY THE LOWEST POWER OF r PRESENT.

E.G.: $A_{n+2} = 3A_{n+1} - 2A_n \xrightarrow{①} r^{n+2} = 3r^{n+1} - 2r^n$
$\xrightarrow{②} r^{n+2} - 3r^{n+1} + 2r^n = 0$
$\xrightarrow{③} \underbrace{r^2 - 3r + 2}_{\text{CHARACTERISTIC POLYNOMIAL}} = 0$

(c) IN A SIMPLE LINEAR RECURRENCE LIKE $B_{n+1} = 2 \cdot B_n$, WE SEE

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MULTIPLY VALUE BY 2

EXPONENTIAL BEHAVIOR — SO WE MIGHT EXPECT EXPONENTIALS TO BE INVOLVED IN THE CLOSED FORMULAE FOR THEIR TERMS.

I.E., DIRECT FORMULAE, WITHOUT RECURSION

(d) TO FIND A CLOSED FORMULA FOR THE TERMS OF A SEQUENCE DEFINED BY A LINEAR RECURRENCE RELATION:

(i) FIND THE ROOTS OF THE CORRESPONDING CHARACTERISTIC POLYNOMIAL; THE ROOTS TELL US THE BASES FOR EXPONENTIAL TERMS WE'LL NEED.

E.G.: • $r^2-3r+2=0 \Rightarrow r=1,2$, SO WE NEED 1^n & 2^n

• $(r+1)(r+2)(r-3)=0 \Rightarrow r=-1,-2,3$, SO WE NEED $(-1)^n$, $(-2)^n$, & 3^n

* WHEN WE HAVE REPEATED ROOTS, WE DON'T REPEAT THE EXPONENTIAL TERMS — WE MODIFY THE DUPLICATES BY MULTIPLYING BY n, THEN n^2, ETC.

E.G.: • $r^2-4r+4=0 \Rightarrow r=2,2$, SO WE NEED 2^n & $n \cdot 2^n$

• $(r+3)^3=0 \Rightarrow r=-3,-3,-3$, SO WE NEED $(-3)^n$, $n\cdot(-3)^n$, & $n^2\cdot(-3)^n$

(ii) THE CHARACTERISTIC POLYNOMIAL TELLS US WHAT BUILDING BLOCKS WE NEED, BUT NOT HOW MUCH OF EACH — THE INITIAL VALUES ALLOW US TO DETERMINE THIS.

E.G., IF $A_{n+2}=5A_{n+1}-4A_n$, $A_0=3$, AND $A_1=7$:

THE CHARACTERISTIC POLYNOMIAL GIVES US $r=-1,4$,

SO $A_n = C(-1)^n + D\cdot 4^n$; TO FIND C & D:

$$\left.\begin{aligned} 3 = A_0 &= C\cdot 1 + D\cdot 1 = C+D \\ 7 = A_1 &= C\cdot(-1) + D\,4 = -C+4D \end{aligned}\right\} \text{SOLVING, } D=2 \ \& \ C=1$$

$\therefore A_n = 1(-1)^n + 2\cdot 4^n$

3. IF $A_0=1$ AND FOR $n\geq 1$, $A_{n+1}=(n+1)A_n$,

THEN $A_0=1$, $A_1=1\cdot 1=1$, $A_2=2\cdot 1$, $A_3=3\cdot 2\cdot 1$, $A_4=4\cdot 3\cdot 2\cdot 1$, ETC,

SO $A_n = n\cdot(n-1)\cdot \cdots \cdot 2\cdot 1 = n!$

4. (a) $B_0 = 5$ AND FOR $n \geq 0$, $B_{n+1} = B_n$

ALL TERMS THE SAME! 5, 5, 5, ... BUT LET'S USE THE SYSTEM TO WARM UP

$B_{n+1} = B_n \rightsquigarrow r^{n+1} = r^n \rightsquigarrow r^{n+1} - r^n = 0 \rightsquigarrow \underbrace{r-1}_{\text{CHARACTERISTIC POLYNOMIAL}} = 0$, SO $r = 1$.

$\therefore B_n = A \cdot 1^n = A$. BUT USING INITIAL TERM, $5 = B_0 = A$, SO $A = 5$.

THUS $B_n = 5$

(b) $C_0 = 3$ AND FOR $n \geq 0$, $C_{n+1} = 10\, C_n$

$C_{n+1} = 10\, C_n \rightsquigarrow r^{n+1} = 10 r^n \rightsquigarrow r^{n+1} - 10 r^n = 0 \rightsquigarrow \underbrace{r - 10}_{\text{CHARACTERISTIC POLYNOMIAL}} = 0$, SO $r = 10$.

$\therefore C_n = A \cdot 10^n$. USING INITIAL TERM, $3 = C_0 = A \cdot 10^0 = A$, SO $A = 3$.

THUS $C_n = 3 \cdot 10^n$.

(c) $D_0 = 0$, $D_1 = 1$, AND FOR $n \geq 0$, $D_{n+2} = 3D_{n+1} - 2D_n$

$$
\begin{aligned}
D_{n+2} = 3D_{n+1} - 2D_n &\rightsquigarrow r^{n+2} = 3r^{n+1} - 2r^n \\
&\rightsquigarrow r^{n+2} - 3r^{n+1} + 2r^n = 0 \\
&\rightsquigarrow \underbrace{r^2 - 3r + 2}_{\text{CHARACTERISTIC POLYNOMIAL}} = 0 \\
&\rightsquigarrow (r-2)(r-1) = 0, \text{ SO } r = 2, 1.
\end{aligned}
$$

$\therefore D_n = A \cdot 2^n + B \cdot 1^n$.

USING INITIAL TERMS: $\left.\begin{aligned} 0 = D_0 &= A \cdot 1 + B \cdot 1 \ \therefore\ A + B = 0 \\ 1 = D_1 &= A \cdot 2 + B \cdot 1 \ \therefore\ 2A + B = 1 \end{aligned}\right\}$ $A = 1$, $B = -1$

THUS $D_n = 1 \cdot 2^n - 1 \cdot 1^n = 2^n - 1$.

(d) $E_0 = 1$, $E_1 = 8$, AND FOR $n \geq 0$, $E_{n+2} = 4E_{n+1} - 4E_n$.

$$E_{n+2} = 4E_{n+1} - 4E_n \rightsquigarrow r^{n+2} = 4r^{n+1} - 4r^n$$
$$\rightsquigarrow r^{n+2} - 4r^{n+1} + 4r^n = 0$$
$$\rightsquigarrow \underbrace{r^2 - 4r + 4}_{\text{CHARACTERISTIC POLYNOMIAL}} = 0$$
$$\rightsquigarrow (r-2)^2 = 0, \text{ so } r = 2, 2.$$

* REPEATED ROOT!

$\therefore E_n = A \cdot 2^n + B \cdot n 2^n$.

USING INITIAL TERMS: $1 = E_0 = A \cdot 1 + B \cdot 0 \quad \therefore A = 1$

$8 = E_1 = A \cdot 2 + B \cdot 2 \quad \therefore 2A + 2B = 8$

$\left.\right\}$ $A = 1$, $B = 3$

THUS $E_n = 1 \cdot 2^n + 3n \cdot 2^n = (1+3n) \cdot 2^n$.

(e) $F_0 = 0$, $F_1 = 1$, AND FOR $n \geq 0$, $F_{n+2} = F_{n+1} + F_n$.

$$F_{n+2} = F_{n+1} + F_n \rightsquigarrow r^{n+2} = r^{n+1} + r^n$$
$$\rightsquigarrow r^{n+2} - r^{n+1} - r^n = 0$$
$$\rightsquigarrow r^2 - r - 1 = 0 \quad \text{so} \quad r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

QUADRATIC FORMULA! $ax^2+bx+c=0 \Rightarrow x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$\therefore F_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

USING INITIAL TERMS: $0 = F_0 = A \cdot 1 + B \cdot 1 \quad \therefore A + B = 0$, so $B = -A$ (*)

$$1 = F_1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right)$$
$$= A\left(\frac{1+\sqrt{5}}{2}\right) - A\left(\frac{1-\sqrt{5}}{2}\right)$$
$$= A\left[\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right] = A\sqrt{5} \quad \therefore A = \frac{1}{\sqrt{5}}$$

$\therefore B = -\frac{1}{\sqrt{5}}$

THUS $F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$

A BIT REMARKABLE, AT FIRST GLANCE, THAT THIS IS ALWAYS A WHOLE NUMBER!

(NOTE THAT $\frac{1+\sqrt{5}}{2} = \phi$, THE "GOLDEN RATIO", AND $\frac{1-\sqrt{5}}{2} = \frac{1}{\phi} = 1 - \phi$)

5. $T(1)=1, \quad T(n) = T(\frac{n}{2}) + 1$

(a)
$$\begin{aligned}
T(1024) &= T(512)+1 = 11\\
T(512) &= T(256)+1 = 10\\
T(256) &= T(128)+1 = 9\\
T(128) &= T(64)+1 = 8\\
T(64) &= T(32)+1 = 7\\
T(32) &= T(16)+1 = 6\\
T(16) &= T(8)+1 = 5\\
T(8) &= T(4)+1 = 4\\
T(4) &= T(2)+1 = 3\\
T(2) &= T(1)+1 = 2
\end{aligned}$$

$\therefore T(\overset{2^{10}}{1024}) = 11 = 1 + \log 1024$

IN GENERAL, $\underbrace{T(2^k) = 1 + \log(2^k)}_{(*)}$

(b) SUBSTITUTING $n=2^k$ IN $(*)$ GIVES $T(n) = 1 + \log n$ FOR $n=2^k$

6. $T(1)=0, \quad T(n) = 2T(\frac{n}{2}) + n$

(a)
$$\begin{aligned}
T(64) &= 2T(32) + 64 = 2\cdot5\cdot32+64 = 6\cdot64\\
T(32) &= 2T(16) + 32 = 2\cdot4\cdot16+32 = 5\cdot32\\
T(16) &= 2T(8) + 16 = 2\cdot3\cdot8+16 = 4\cdot16\\
T(8) &= 2T(4) + 8 = 2\cdot8+8 = 3\cdot8\\
T(4) &= 2T(2) + 4 = 2\cdot2+4 = 2\cdot4\\
T(2) &= 2T(1) + 2 = 2 = 1\cdot2
\end{aligned}$$

② (NOTE THAT EACH IS A CERTAIN MULTIPLE OF 2^k... SPECIFICALLY $k\cdot2^k$)

(b) IN GENERAL, $T(2^k) = k\cdot 2^k$, SO IF $n=2^k$, $T(n) = (\log n)\cdot n = n\log n$.
$(\log n = k)$

7. $T(1)=0, \quad T(n) = 2T(\frac{n}{2}) + n^2$

(a)
$$\begin{aligned}
T(64) &= 2T(32) + 64^2 = 2^7+2^8+2^9+2^{10}+2^{11}+2^{12} = 2(2^{12}-2^6)\\
T(32) &= 2T(16) + 32^2 = 2^6+2^7+2^8+2^9+2^{10} = 2(2^{10}-2^5)\\
T(16) &= 2T(8) + 16^2 = 2^5+2^6+2^7+2^8 = 2(2^8-2^4)\\
T(8) &= 2T(4) + 8^2 = 2^4+2^5+2^6 = 2(2^6-2^3)\\
T(4) &= 2T(2) + 4^2 = 2^3+2^4 = 2(2^4-2^2)\\
T(2) &= 2T(1) + 2^2 = 2^2 = 2(2^2-2^1)
\end{aligned}$$

JUST CONVENIENT FORMS FOR PART (b) — EVEN T(2) FITS THE PATTERN, USING:

$$2^A + 2^{A+1} + \cdots + 2^B = 2^{B+1} - 2^A$$

(b) IN GENERAL $T(2^k) = 2^{k+1} + 2^{k+2} + \cdots + 2^{2k} = 2(2^{2k} - 2^k)$

SO $T(n) = 2(n^2 - n)$